Hydrodynamic Coefficients of an Elongated Body Rapidly Approaching a Free Surface

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A hydrodynamic analysis is made for the motion of a submerged, elongated body of revolution rapidly approaching a free-surface with longitudinal, transverse, and rotational velocities, and an arbitrary angle of attack. Axial doublet distributions are used to generate the flowfield and the prescribed body shape. The hydrodynamic forces and moments acting on the body are determined by the generalized Lagally theorem. Expressions for the hydrodynamic force and moment are derived and numerical solutions for the hydrodynamic coefficients are obtained. Through the characteristic behavior of these coefficients, effects of the free surface on the body dynamics are examined.

I. Introduction

FOR a body moving near a free surface, studies have shown that it experiences hydrodynamic forces and moments induced by the free surface. Most of the literature dealing with effects of a free surface on the dynamics of fully submerged bodies appeared between 1950 and the early 1960's. These were notably the works of Pond,[1] Havelock,[2] Breslin and Kaplan,[3] Woo,[4] Goodman,[5] Clark and Robertson,[6] Martin,[7] and Moran.[8,9] There have also been a number of analyses on the related subject of free surface effects on the added masses of moving bodies of Eisenberg,[10] Motora,[1] Bottaccini,[12] Waugh and Ellis,[13] and more recently by Garrison and Berklite.[14] For a body moving at a constant speed and high Froude numbers toward a free surface at a right angle, both analytical[3] and experimental[6] studies reported that, at a certain distance from the free surface, the body undergoes an acceleration which increases as the depth of submergence decreases. This implies that the hydrodynamic resistance of the body decreases upon approaching a free surface. Such a tendency is consistent with the analyses of Ensenberg[10] and Motora,[11] in which the added masses of the body were shown to decrease monotonically with the submergence depth.

In the aforementioned references, the motion of the body was restricted to a single transverse or longitudinal translation. This could be either normal or parallel to the free surface, or a single rotation about one of the body axes. In such cases, the hydrodynamic forces and moments consist of two parts; one of which is proportional to the body acceleration and the other to the square of the body velocity. However, for a body moving near a free surface with 2 or more degrees-of-freedom, due to interactions of the velocity components of the body, there exist additional hydrodynamic forces and moments. These cannot be deduced from results for motions of 1 degree-of-freedom.

The purpose of this paper is to examine free surface effects on the hydrodynamics of a body of revolution moving at high Froude numbers with longitudinal, transverse, and rotational velocities in a time-varying, potential field. The work consists of 3 major parts. In the first, a generating axial singularity distribution is determined by a refinement of formulas due to von Karman[15] and Munk.[16] In the second, the generalized Lagally theorem[17,18] is applied to derive expressions for the forces and moments. In the last, the hydrodynamic force and moment coefficients are evaluated. Numerical results presented in this paper were obtained in 1970-71 at the Westinghouse Research Laboratories.

II. Formulation of Problem

We shall consider a submerged, elongated body of revolution moving rapidly near a free surface of an inviscid, incompressible fluid. Except for the free surface, all other boundaries are assumed to be distant, so that their effects are negligible. A coordinate system (X,Y,Z) is fixed at the free surface with the X-Y plane representing its undisturbed level, and the Z-axis pointing upwards, Fig. 1. A second coordinate system (x,y,z) is attached to the center of mass $C(X_C, Y_C, Z_C)$ of the moving body, with the z-axis as the axis of symmetry. In this coordinate system the equation of the body will be expressed as $r(z)=(x^2+y^2)^{1/2}$. We shall formulate the problem for planar motion. For convenience, the x-z plane is assumed to be parallel to the fixed X-Z plane. The body has linear velocities U and W in the x and z directions, respectively, and angular velocity a about the y axis.

The boundary condition on the surface of the body is

$$\frac{\partial}{\partial n}\Phi(x,y,z,t)=\bar{n}\cdot\bar{V}, \tag{1a}$$

$$\bar{V}=\bar{e}_1U(t)+\bar{e}_3W(t)+\bar{e}_2q(t)\times\bar{R}_c \tag{1b}$$

where Φ is the total velocity potential; $\bar{V}$ and $\bar{R}_c$ are the absolute velocity and the position vector of a point (x,y,z) on the body surface, respectively; $\bar{n}$ is the unit outward surface normal; $\bar{e}_1$, $\bar{e}_2$, and $\bar{e}_3$ are the unit vectors in the x, y, and z directions, respectively; and t denotes time.

The linearized Bernoulli equation at the free surface, in the fixed frame of reference is

$$\frac{\partial}{\partial t}\Phi+g\zeta(X,Y,t)=0 \tag{2}$$

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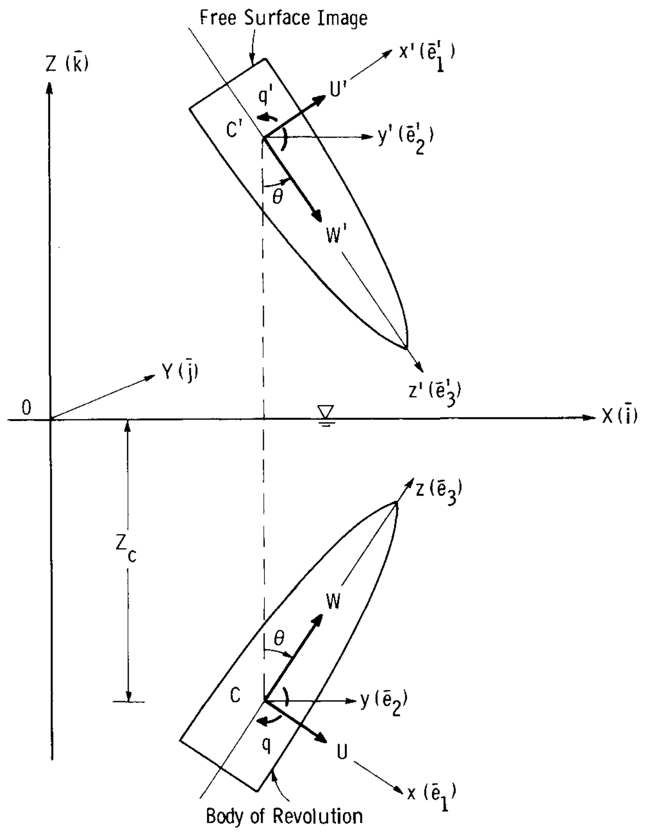


Fig. 1 Definition sketch.

where g is the gravitational potential, and $Z=\zeta(X,Y,t)$ is the equation of the disturbed free surface. In the present study, the body is assumed to be moving at a high speed, at which the gravitational term in Eq. (2) may be neglected in comparison with the inertia term. Integration of the resulting equation with respect to time then yields

$$\Phi(X,Y,0,t)=f(X,Y)$$

or, since $f(X,Y)$ is independent of time, and the free surface was initially undisturbed,

$$\Phi(X,Y,0,t)=0 \tag{3}$$

Condition (3) can be satisfied by introducing a mirror image of the body in the free surface, but with the instantaneous linear and angular velocities of the body in the opposite sense from that of an optical mirror image. Thus, in a left-handed coordinate system (x', y', z'), which is a mirror image of the (x,y,z) system, Fig. 1), the velocities U', W', and q' of the image are defined by

$$U'=-U,\ W'=-W,\ q'=-q$$

Note that the positive sense of rotation of the angular velocities q' about the y' axis is that of the left-hand screw. The direction cosines of the 3 coordinate systems (X,Y,Z), (x,y,z), and (x', y', z') are given in Table 1 where $\bar{i},\bar{j}$, and $\bar{k}$ are unit vectors in the X, Y, and Z directions, respectively; $\bar{e}_1'$, $\bar{e}_2'$, and $\bar{e}_3'$ are unit vectors in the x', y', and z' directions, respectively; and $a=Z_c/\cos\theta$, $(Z_c \leqq 0)$. From the table we obtain

$$x'=x\cos 2\theta+z\sin 2\theta+2Z\sin\theta \tag{4a}$$

$$z'=x\sin 2\theta-z\cos 2\theta-2Z_c\cos\theta \tag{4b}$$

and the point $(0,0,z')$ has the (x,y,z) coordinates

$$(z'\sin 2\theta+2Z_c\sin\theta, 0, -z'\cos 2\theta-2Z_c\cos\theta) \tag{5}$$

A. Velocity Potentials and Doublet Distributions

We shall seek a solution of the Laplace equation such that the boundary condition Eq. (1) on the body, and its equivalent on the image, are satisfied. We shall begin with an axial doublet distribution which gives a good approximation to the flow about the body moving in an unbounded fluid. This distribution will then be modified so as to correct the boundary condition on the body for the disturbance due to the same distribution, but of opposite sign, within the image.

First Approximation – Unbounded Fluid

Let Φ_1 (x,y,z,t) denote the total velocity potential for the body in planar motion at a great distance from the free surface. This may be written in the form

$$\Phi(x,y,z,t)=U(t)\Phi_1(x,y,z)+W(t)\Phi_3(x,y,z)+q(t)\Phi_5(x,y,z) \tag{6}$$

where Φ_1, Φ_3, and Φ_5 are potentials due to unit values of U, W, and q, respectively. We shall assume that, for an elongated body of revolution, these potentials can be generated by distributions of doublets along the axis of symmetry. Their strengths, given by the slender-body approximations of von Karman,[15] and Munk,[16] would be

$$\bar{m}_0(z)=r^2(z)\left[\tfrac{1}{2}U\bar{e}_1+\tfrac{1}{4}W\bar{e}_3+\tfrac{1}{2}qz\bar{e}_1\right] \tag{7}$$

Refinements by Weinblum[19] and Lanbweber[20] assume

$$\bar{m}_1(z)=r^2(z)\left(UC_1\bar{e}_1+WC_3\bar{e}_3+qzC_5\bar{e}_1\right) \tag{7a}$$

where the C_i are constants. As was shown in Ref. 20, explicit forms of C_1 and C_3 can be obtained by applying the added-mass formulas of Taylor[21] and Landweber,[22] which relate the added masses to the generating singularities. Although this added-mass formula does not apply to pure rotation, an approximate version of the formula, given in Ref. 17, which is exact for spheroids, has been used to obtain C_5 (see Appendix 1). These formulas are

$$(1+k_T)V=4\pi\int_{z1}^{z2} m_T dz \tag{8}$$

$$(1+k_L)V=4\pi\int_{z1}^{z2} m_L dz \tag{9}$$

$$(1+k_R)\int_V (x^2+z^2)dV \doteq 4\pi\int_{z1}^{z2} z m_R dz \tag{10}$$

where k_T and k_L are the added-mass coefficients for transverse and longitudinal motions; k_R is the added moment-of-inertia coefficient for rotation about the y-axis; m_T, m_L, and m_R are doublet distributions corresponding to unit values of

Table 1 Direct cosines for coordinate transformation

	$\bar{e}_1,x$	$\bar{e}_2,y$	e_3,z		$\bar{e}_1,x$	$\bar{e}_2,y$	$\bar{e}_3,z+a$
$\bar{i},X-X_c$	$\cos\theta$	0	$\sin\theta$	$\bar{e}_1',x'$	$\cos 2\theta$	0	$\sin 2\theta$
$\bar{j},Y$	0	1	0	$\bar{e}_2',y'$	0	1	0
$\bar{k},Z-Z_c$	$-\sin\theta$	0	$\cos\theta$	$\bar{e}_3',z'+a$	$\sin 2\theta$	0	$-\cos 2\theta$

Table 2 Comparison of doublet distributions

	m_L/W	m_T/U	m_R/qz
Exact	$0.007571\left(1-\frac{z^2}{c^2}\right)$	$0.01389\left(1-\frac{z^2}{c^2}\right)$	$0.01342\left(\frac{z^2}{c^2}\right)$
Munk Approx.	$0.00741\left(1-\frac{z^2}{a^2}\right)$	$0.01429\left(1-\frac{z^2}{a^2}\right)$	$0.01429\left(1-\frac{z^2}{a^2}\right)$
Refined Approx.	$0.007465\left(1-\frac{z^2}{a^2}\right)$	$0.01367\left(1-\frac{z^2}{a^2}\right)$	$0.01295\left(1-\frac{z^2}{a^2}\right)$

U, W, and q, respectively; $\mathcal{V}$ is the volume of the body, and z_1, z_2, are the z-coordinates of the body extremities. Evaluating Eqs. (8), (9), and (10) with the aid of Eq. (7a) then yields

$$\vec{m}_1(z) = \tfrac{1}{4}r^2(z)\left[(\lambda_T U+\lambda_R zq)\bar{e}_1+\lambda_L W\bar{e}_3\right]$$

$$\lambda_T = 1+k_T,\ \lambda_L 1+k_L,\ \lambda_R = 1+k_R' \tag{11}$$

$$1+k_R' = (1+k_R)\left[1+\frac{1}{4}\frac{\int_{z1}^{z2} r^4 dz}{\int_{z1}^{z2} z^2r^2 dz}\right]$$

The potentials due to these axial doublet distributions are then

$$\phi_1(x,y,z) = -\frac{1+k_T}{4}\int_{z1}^{z2}\frac{xr^2(\zeta)d\zeta}{[x^2+y^2+(z-\zeta)^2]^{3/2}} \tag{12}$$

$$\phi_3(x,y,z) = -\frac{1+k_L}{4}\int_{z1}\frac{(z-\zeta)r^2(\zeta)d\zeta}{[x^2+y^2+(z-\zeta)^2]^{3/2}} \tag{13}$$

$$\phi_5(x,y,z) = -\frac{1+k_R'}{4}\int_{z1}^{z2}\frac{x\zeta r^2(\zeta)d\zeta}{[(x^2+y^2+(z-\zeta)^2]^{3/2}} \tag{14}$$

One can judge the improvement of approximation Eq. (11) over Eq. (7) by comparing these in Table 2 with the exact doublet distributions for the spheroid

$$\frac{z^2}{a^2}+\frac{r^2}{b^2}=1$$

with $a/b=6$, $c=(a^2-b^2)^{1/2}=1$, and hence $a=1.0142$.

One sees that the refined approximation gives a significant improvement over the Munk approximation not only at $z=0$ but also at other values of z, since the curves for the refined nd exact distributions intersect in the range $-c<z<c$.

Since the approximate distribution extends slightly beyond the foci of the spheriod, one may interpret the results it yields as corresponding to a slightly longer body than the given one. For bodies with a spheroidlike nose, a well-known rule-of-thumb[20] is that the body extends beyond the distribution by about half the radius of curvature at that end. For finer or blunter ends, larger or smaller fractions of the radius of curvature are required. This suggests the calculations be terminated when the nose of the body is at half its radius of curvature from the free surface.

Second Approximation

Let $\Phi_2(x', y', z', t)$ denote the total velocity potential for the mirror image, analogous to Φ_1, which will be written as

$$\Phi_2 = -U\phi_1' - W\phi_3' - q\phi_5' \tag{15}$$

where ϕ_1', ϕ_3', ϕ_5' are the unit potentials. These are given by Eqs. (12), (13) and (14), with (x,y,z) replaced by (x', y', z'). Substituting the expressions (4) for x', and z', then replacing ζ with z', we obtain

$$\Phi_2 = \int_{z_1'}^{z_2'}\frac{r^2(z')}{4R^3}\left[(\lambda_T U+\lambda_R qz')(x\cos 2\theta + z\sin 2\theta + 2Z_c\sin\theta) + \lambda_L W(x\sin 2\theta - z\cos 2\theta - 2Z_c\cos\theta - z')\right]dz' \tag{16}$$

where

$$R^2 = x^2-2x(z'\sin 2\theta+2Z_c\sin\theta)+z^2+2zz'\cos 2\theta+z'^2 + 4Z_c(z+z')\cos\theta+4Z_c^2 \tag{17}$$

We shall now correct the doublet distributions within the body so as to compensate for the velocity field due to the image potential Φ_2. Put

$$u = \left.\frac{\partial\Phi_2}{\partial x}\right|_{(0,0,z)},\quad w = \left.\frac{\partial\Phi_2}{\partial z}\right|_{(0,0,z)}$$

A corrected doublet distribution $\vec{m}(z)$ can then be derived by replacing U by $U-u$ and W by $W-w$ in Eq. (11). Thus we obtain

$$\vec{m}(z) = \Delta_1(z)\bar{e}_2+\Delta_3(z)\bar{e}_3 \tag{18}$$

where

$$\Delta_1(z) = \tfrac{1}{4}r^2(z)\left[(1+k_T)(U-u)+(1+k_R')zq\right], \tag{19a}$$

$$\Delta_3(z) = \frac{1+k_L}{4}r^2(z)(W-w) \tag{19b}$$

where, by Eqs. (16) and (17), we have

$$u = \int_{z_1'}^{z_2'}\left\{\frac{r^2(z')}{4R_0^3}\left[(\lambda_T U+\lambda_T U+\lambda R qz')\cos 2\theta+\lambda_L W\sin 2\theta\right] + \frac{3r^2(z')}{4R_0^5}\left[(\lambda_T U+\lambda_R qz')(z\sin 2\theta+2Z_c\sin\theta) - \lambda_L W(z\cos 2\theta+2Z_c\cos\theta+z')\right](z'\sin 2\theta+2Z_c\sin\theta)\right\}dz' \tag{20}$$

$$w = \int_{z_1'}^{z_2'}\left\{\frac{r^2(z')}{4R_0^3}\left[(\lambda_T U+\lambda_R qz')\sin 2\theta-\lambda_L W\cos 2\theta\right] - \frac{3r^2(z')}{4R_0^5}\left[(\lambda_T U+\lambda_R qz')(z\sin 2\theta+2Z_c\sin\theta) - \lambda_L W(z\cos 2\theta+2Z_c\cos\theta+z')\right] \times (z+z'\cos 2\theta+2Z_c\cos\theta)\right\}dz' \tag{21}$$

with

$$R_0^2 = z^2 + 2zz'\cos 2\theta + z'^2 + 4Z_c(z+z')\cos\theta + 4Z_c^2 \quad (22)$$

Here R_0 is the distance from a point on the axis of the body to a point on the axis of its image.

B. Hydrodynamic Force and Moment Formulas

The generalized Lagally theorem[17,18] will now be applied to determine the hydrodynamic force and moment acting on the body. This theorem expresses the force in terms of the singularity distributions within the body. In the present case, when the singularities consist of doublet distributions, the formula becomes

$$\bar{F} = \rho V \frac{\mathrm{d}\bar{V}^*}{\mathrm{d}t} - 4\pi\rho \frac{\mathrm{d}}{\mathrm{d}t}\int_{z_1}^{z_2} \bar{m}(z)\,\mathrm{d}z - 4\pi\rho\int_{z_1}^{z_2}(\Delta_1 \frac{\partial}{\partial x} + \Delta_3 \frac{\partial}{\partial z})\ (u\bar{e}_1 + w\bar{e}_3)\,\mathrm{d}z \quad (23)$$

where

$$\bar{V}^* = (U+qz^*)\bar{e}_1 + W\bar{e}_3 \quad (24)$$

is the velocity of the centroid, at $(0,0,z^*)$. The absolute acceleration $\mathrm{d}\bar{V}/\mathrm{d}t$ can be expressed in the form

$$\frac{\mathrm{d}\bar{V}^*}{\mathrm{d}t} = (\dot{U} + \dot{q}z^* + qW)\bar{e}_1 + (\dot{W} - qU - q^2 z^*)\bar{e}_3 \quad (25)$$

in which the dot indicates differentiation with respect to time. Also, we have

$$\frac{\mathrm{d}\bar{m}}{\mathrm{d}t}\ (\dot{\Delta}_1 + q\Delta_3)\bar{e}_1 + (\dot{\Delta}_3 - q\Delta_1)\bar{e}_3 \quad (26)$$

Hence, with $\bar{F} = \bar{e}_1 f_1 + \bar{e}_3 f_3$, we have

$$f_1 = \rho \mathcal{V}(\dot{U} + \dot{q}z^* + qW) - 4\pi\rho\int_{z_1}^{z_2} [\dot{\Delta}_1 + \Delta_1 u_x + \Delta_3(q+u_z)]\,\mathrm{d}z \quad (27)$$

$$f_3 = \rho \mathcal{V}(\dot{W} - qU - q^2 z^*) - 4\pi\rho\int_{z_1}^{z_2} [\dot{\Delta}_3 + \Delta_3 w_z + \Delta_1(w_x - q)]\,\mathrm{d}z \quad (28)$$

where

$$u_x = \frac{\partial^2\Phi_2}{\partial x^2}\Big|_{(0,0,z)},\quad w_z = \frac{\partial^2\Phi_2}{\partial z^2}\Big|_{(0,0,z)},$$

$$u_z = w_x = \frac{\partial^2\Phi_2}{\partial x\partial z}\Big|_{(0,0,z)} \quad (29)$$

Substitution of Eq. (19) in Eq. (27) now gives

$$\begin{aligned} f_1 &= \rho \mathcal{V}(\dot{U} + z^*\dot{q} + qW) - \pi\rho\int_{z_1}^{z_2} r^2(z) \\ &\times [(1+k_T)\dot{U} + (1+k_R')z\dot{q} + (1+k_L)qW \\ &- \lambda_T\dot{u} + u_x(\lambda_T U + \lambda_R qz) \\ &- \lambda_L wq + \lambda_L W u_z - \lambda_T u u_x - \lambda_L w u_z]\,\mathrm{d}z \end{aligned} \quad (30)$$

Since

$$\pi\int_{z_1}^{z_2} r^2(z)\,\mathrm{d}z = \mathcal{V},\quad \pi\int_{z_1}^{z_2} z r^2(z)\,\mathrm{d}z = \mathcal{V}z^*$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\, u(U,W,q,Z_c,\theta) = u(\dot{U},\dot{W},\dot{q},Z_c,\theta) + \dot{Z}_c u_z + q u_\theta \quad (31)$$

where

$$\dot{Z}_c = -U\sin\theta + W\cos\theta, u_z = \partial u/\partial Z_c, u_\theta = \partial u/\partial\theta \quad (32)$$

expression (30) for f_1 reduces to

$$\begin{aligned} f_1 &= -\rho \mathcal{V}(k_T\dot{U} + k_L qW + k_R' z^*\dot{q}) \\ &+ \pi\rho\int_{z_1}^{z_2} r^2(z)\,[\lambda_T u(\dot{U},\dot{W},\dot{q}) + \lambda_T \dot{Z}_c u_z \\ &+ \lambda_T q u_\theta - u_x(\lambda_T U + \lambda_R qz) \\ &+ \lambda_L wq - \lambda_L W u_z + \lambda_T u u_x + \lambda_L w w_x]\,\mathrm{d}z \end{aligned} \quad (33)$$

Similarly, we obtain from Eqs. (28) and (19)

$$\begin{aligned} f_3 &= -\rho \mathcal{V}(k_L\dot{W} - k_T qU - k_R' z^* q^2) \\ &+ \pi\rho\int_{z_1}^{z_2} r^2(z)\,[\lambda_L w(\dot{U},\dot{W},\dot{q}) + \lambda_L \dot{Z}_c w_z + \lambda_L q w_\theta \\ &- \lambda_L W w_z - u_z(\lambda_T U + \lambda_R qz) \\ &- \lambda_T qu + \lambda_T u u_x + \lambda_L w w_z]\,\mathrm{d}z \end{aligned} \quad (34)$$

In these expressions for the force components, the integral represents the effect of the free surface.

The moment about the y-axis through the center of gravity, $\bar{M}_2 = \bar{e}_2 M_2$, can also be expressed in terms of the doublet distributions, except for an added moment of inertia due to rotation about the y-axis, A_{55}. Although the total added moment of inertia due to rotation depends upon Z_c and θ, the part of it designated by A_{55} may be considered as constant to the order of approximation of the present treatment, and may be taken as given by

$$A_{55} = k_R B_{55}, B_{55} = \rho\int_{\mathcal{V}}(x^2+z^2)\,\mathrm{d}\tau = \rho\,\mathcal{V} r_G^2 \quad (35)$$

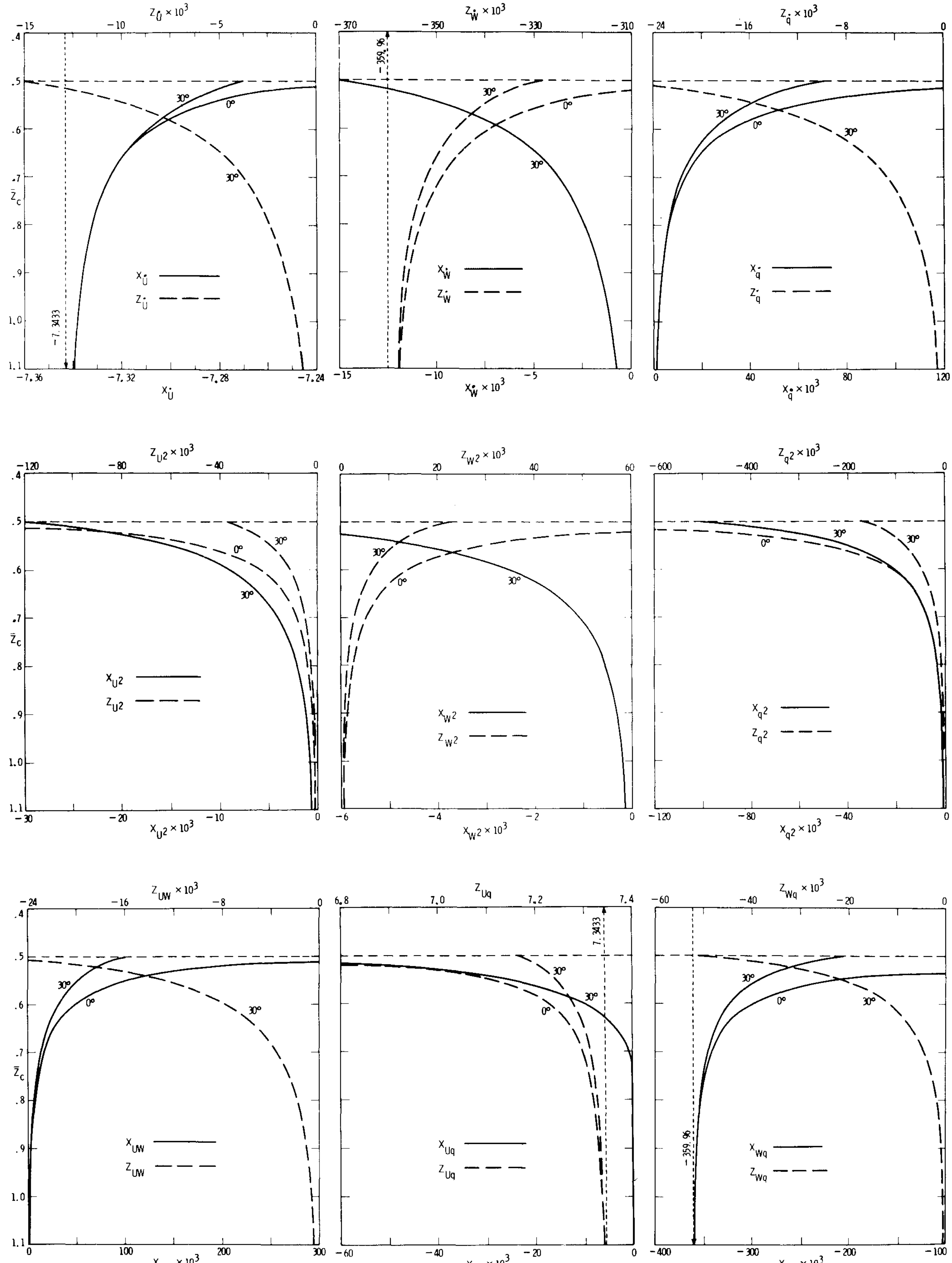
where B_{55} is the moment of inertia of the body, of the same density as the fluid, and r_G the radius of gyration about the y axis.

The expression for the moment, in the form given by Eq. (64) of Ref. 17 will now be applied. This gives

$$\begin{aligned} M_2 &= -\dot{q}A_{55} + \rho \mathcal{V} z^*(\dot{U} + qW) \\ &- 4\pi\rho\frac{\mathrm{d}}{\mathrm{d}t}\int_{z_1}^{z_2} m_Q(U-u)\,\mathrm{d}z - 4\pi\rho\int_{z_1}^{z_2}[\Delta_1(W-w) \\ &- \Delta_3(U-u) + z(\Delta_1 u_x + \Delta_3 u_z)]\,\mathrm{d}z \end{aligned} \quad (36)$$

where m_Q is the strength of the doublet distribution when the body is rotating with unit angular velocity, but the image body is at rest. Here, consistent with the present approximation, the term of (64) of Ref. 17, $\mu_{oj}v'_{\beta j}$ has been neglected. To the same order of approximation, we may take m_Q to be constant

$$m_Q = \tfrac{1}{4}(1+k_R')zr^2(z) \quad (37)$$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$\bar{Z}_c$
0°
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−7.3433
−359.96
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a)

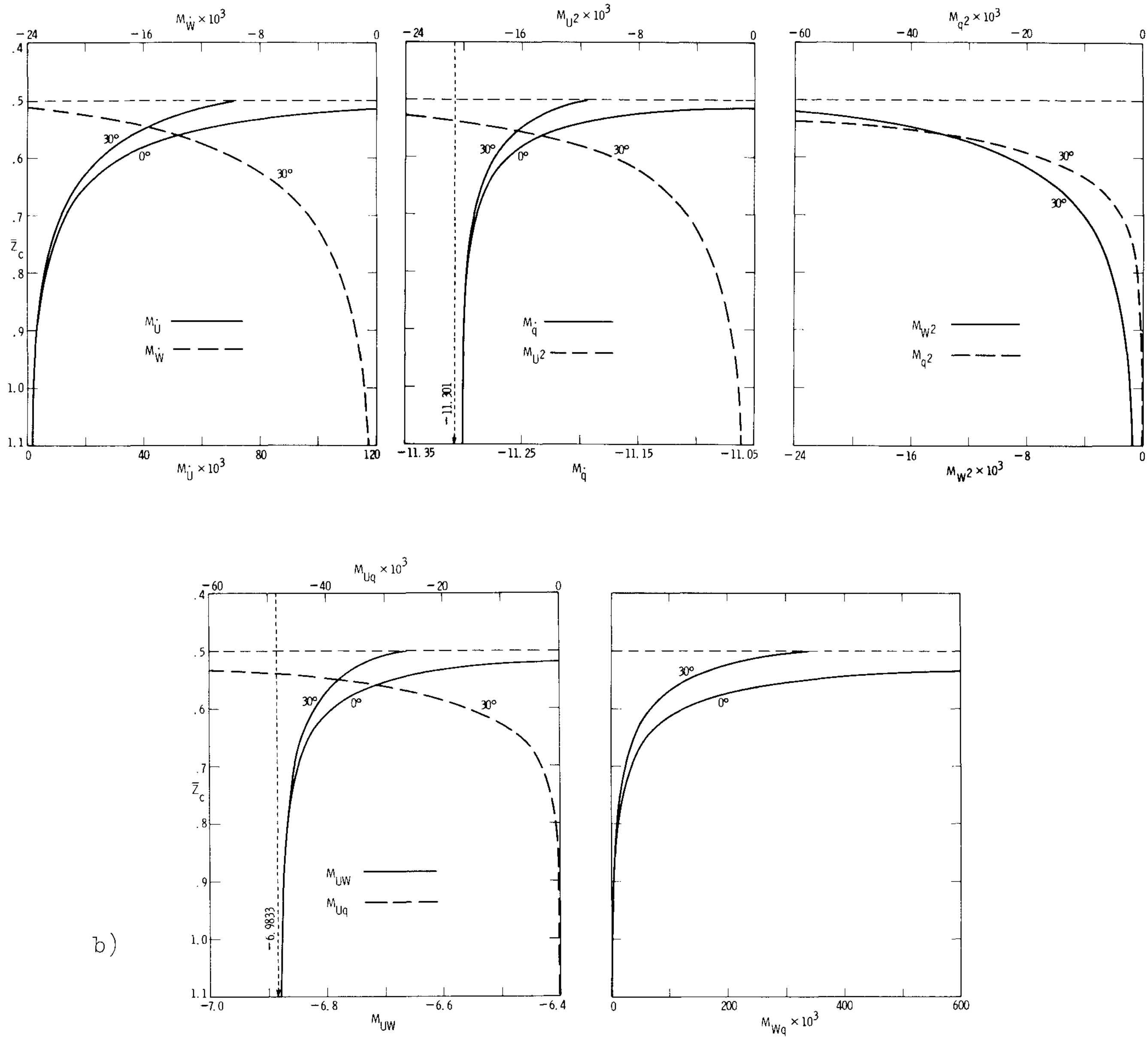

Fig. 2 Hydrodynamic force and moment coefficients for a 6:1 spheroid.

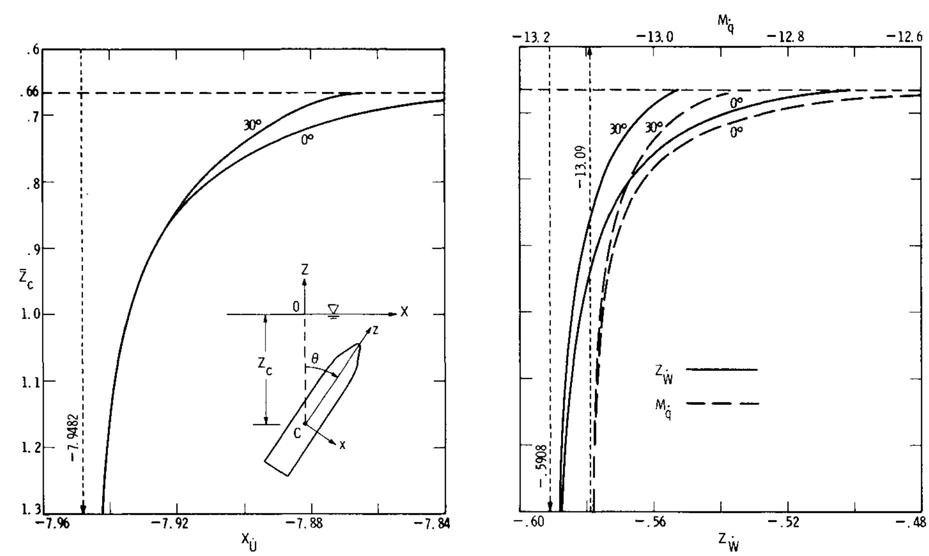

Fig. 3 Hydrodynamic force and moment coefficients for a prescribed body of revolution.

as is seen from Eq. (19). Expression (36) for M_2 then becomes

$$M_2 = -\rho \, \mathcal{V}[k_R r_G^2 \dot{q} + k_R' z^* (\dot{U}+qW) + (k_T - k_L) UW]$$

$$+ \pi\rho \int_{z1}^{z2} r^2(z) \, [\lambda_R z\dot{u} + (k_T - k_L)(Uw + uW - uw)$$

$$+ \lambda_R zqw - zu_x(\lambda_T U - \lambda_T u + \lambda_R zq)$$

$$- \lambda_L zu_z(W-w)] \, dz \tag{38}$$

in which Eqs. (19) and (35) have been applied and $\dot{u}$ is given by Eqs. (31) and (32). Here also the integral represents the effect of the free surface.

C. Hydrodynamic Force and Moment Coefficients

To facilitate the dynamic and trajectory calculations, we can express for force and moment formulas in terms of nondimensional hydrodynamic coefficients and their corresponding kinematic quantities:

$$f_1 = \alpha_1(X_{U2}U^2 + X_{W2}W^2 + X_{UW}UW)$$
$$+ \alpha_2(X_{\dot{U}}\dot{U} + X_{\dot{W}}\dot{W} + X_{Uq}Uq + X_{Wq}Wq)$$
$$+ \alpha_3(X_{\dot{q}}\dot{q} + X_{q2}q^2) \tag{39}$$

$$f_3 = \alpha_1(Z_{U2}U^2 + Z_{W2}W^2 + Z_{UW}UW)$$
$$+ \alpha_2(Z_{\dot{U}} + Z_{\dot{W}}\dot{W} + X_{Uq}Uq + Z_{Wq}Wq)$$
$$+ \alpha_3(Z_{\dot{q}}\dot{q} + Z_{q2}q^2) \tag{40}$$

$$M_2 = \alpha_2(M_{U2}U^2 + M_{W2}W^2 + M_{UW}UW)$$
$$+ \alpha_3(M_{\dot{U}}\dot{U} + M_{\dot{W}}\dot{W} + M_{UQ}Uq + M_{Wq}Wq)$$
$$+ \alpha_4(M_{\dot{q}}\dot{q} + M_{q2}q^2) \tag{41}$$

in which the quantities $X_{U2}, X_{UW}, \ldots$ are nondimensional hydrodynamic force coefficients, and the α_i are the dimensional factors

$$\alpha_i = (\pi/8)\rho D^{i+1}$$

where D is the maximum diameter of the body. Expressions for these coefficients can be obtained by comparing Eqs. (39), (40), and (41) to the fully-expanded forms of Eqs. (33), (34) and (38). Numerical evaluations of the lengthly mathematical expressions obtained for the 27 hydrodynamic coefficients are presented graphically in Sec. III.

III. Numerical Results

Numerical solutions for hydrocoefficients have been obtained on a UNIVAC-1106 for a 6:1 spheroid and for a prescribed body of revolution. The results are presented in Figs. 2 and 3. Each hydrocoefficient is plotted as a function of the nondimensional submergence depth $\bar{Z}_C$ for 2 θ angles. ($\bar{Z}_C$ is defined as $|Z_C|$ divided by the body length). The asymptotic value of a hydrocoefficient for $\bar{Z}_C \to \infty$, if nonzero, is given by a vertical dotted line terminating on the abscissa. Whenever a coefficient is identically zero for $\theta = 0°$ (hence independent of $\bar{Z}_C$), it is omitted from the figures.

In Fig. 2, the 27 hydrocoefficients for a 6:1 spheroid in planar motion near a free surface are presented for the range of $0.5 < \bar{Z}_C \leq 1.1$, where the effect of the free surface is most prominent. Note that when $\theta = 0°$, the value $\bar{Z}_C = 0.5$ corresponds to the position at which the leading edge of the spheroid touches the free surface.§

In Fig. 3, 3 hydrocoefficients are given for a body of revolution with a truncated base.¶ Because of the characteristic variations of the hydrocoefficients of this prescribed body and those of the 6:1 spheroid are similar, only $X_{\dot{U}}$, $Z_{\dot{W}}$, and $M_{\dot{q}}$ are provided as examples.

In Fig. 4, for a 6:1 spheroid at $\theta = 90°$ (i.e., the major axis is parallel to the free surface), the present results are compared with Eisenberg's approximate solutions.[10] k_{11} and k_{33} are the added-mass coefficients for transverse and longitudinal motions, respectively, and k_{55} is the coefficient of added moment-of-inertia about the y axis. (Note that in the limit of $\bar{Z}_C \to \infty$, $k_{11} = K_T$, $k_{33} = k_L$, and $k_{55} = k_R$). Also, k_{11}, k_{33}, and k_{55} can be related to the hydrocoefficients by the following formulas

$$k_{11} = \alpha_2 X_{\dot{U}}/\rho \, \mathcal{V} = -X_{\dot{U}}/8$$

$$k_{33} = -\alpha_2 Z_{\dot{W}}/\rho \, \mathcal{V} = -Z_{\dot{W}}/8$$

$$k_{55} = -\alpha_4 M_{\dot{q}}/I_2 = -5M_{\dot{q}}/74$$

where I_2 is the moment of inertia about the y axis of the body with its mass density replaced by that of the fluid.

IV. Discussion

For a 6:1 spheroid, as $\bar{Z}_C \to \infty$, there exist 6 nontrivial hydrocoefficients, namely $Z_{\dot{U}}$, $Z_{\dot{W}}$, $M_{\dot{q}}$, X_{Wq}, Z_{Uz}, and M_{UW}. Except for Z_{Uq}, they all have negative values. As the spheroid approaches the free surface, some of the hydrocoefficients vary with the submergence depth. As shown in Fig. 2, all coefficients that have nontrivial values vary monotonically with $\bar{Z}_C$; and the absolute values of the deviations from their deep-water values increase as $\bar{Z}_C$ decreases. In the case of $\theta = 30°$, all 27 hydrocoefficients are nontrivial as compared to a total of 6 for $\bar{Z}_C \to \infty$. In the special case of $\theta = 0°$, 13 hydrocoefficients are nontrivial. This indicates that there are hydrodynamic forces and moments which are induced by the free surface.

As an example, we consider a simple case of a 6:1 spheroid moving toward a free surface with a constant velocity W at $\theta = 0°$, and $U = \dot{U} = \dot{W} = q = \dot{q} \equiv 0$. From Eqs. (39, (40), and (41) we have

$$f_1 = 0 \qquad f_3 = \alpha_1 Z_{W2} W^2 \qquad M_2 = 0$$

Figure 2 shows that, near a free surface, Z_{W2} is positive, and its magnitude increases as the spheroid approaches the free surface. Since α_1 and W^2 are positive, f_3 is a positive force exerted on the body in the positive z-direction by the fluid. Physically, f_3 is seen to be a force pulling the spheroid toward the free surface. This particular aspect of free surface effects has already been reported for slender bodies and spheres.[3,6]

As seen in Fig. 3, the characteristic variations of $X_{\dot{U}}$, $Z_{\dot{W}}$ and $M_{\dot{q}}$ for the prescribed body of revolution near a free surface are similar to their counterparts for the 6:1 spheroid. As before, the effects of the free surface are seen to increase as $\bar{Z}_C$ decreases. The percentage variations in $X_{\dot{U}}$, $Z_{\dot{W}}$ and $M_{\dot{q}}$ at $\theta = 0°$ between $Z_C = 0.7$ and $Z_C \to \infty$ are 1%, 9%, and 2%, respectively.

Three of Eisenberg's results for the added mass coefficients for large Froude numbers, namely k_{11}, k_{33} and k_{55}, can be compared with the present results for a 6:1 spheroid at $\theta = 90°$. Though both are approximate solutions, Fig. 4 shows that they agree both in the orders of magnitude and in the characteristic variations of these coefficients near a free surface.

§According to the remark at the end of the Section *First Approximation – Unbounded Fluid*, however, the body generated by the assumed distribution is longer than that given by the distance of the focus of the spheroid from its extremity, 0.007 in the present case. Hence the limiting value of $\bar{Z}_c$ should be taken at $\bar{Z}_C = 0.507$ instead of 0.500.

¶The body shown in Fig. 3 has an overall length of 32.43 ft, and a base diameter of 6.17 ft. The mass center of the body is located on the axis of symmetry at a distance of 10.92 ft from the base; the centroid is at 3.12 ft above the mass center. The total volume of the body is 829.4 ft^3. The equivalent spheroid has a fineness ratio of 4.65, with $k_T = 0.8825$, $k_L = 0.0656$, and $k_R = 0.688$.

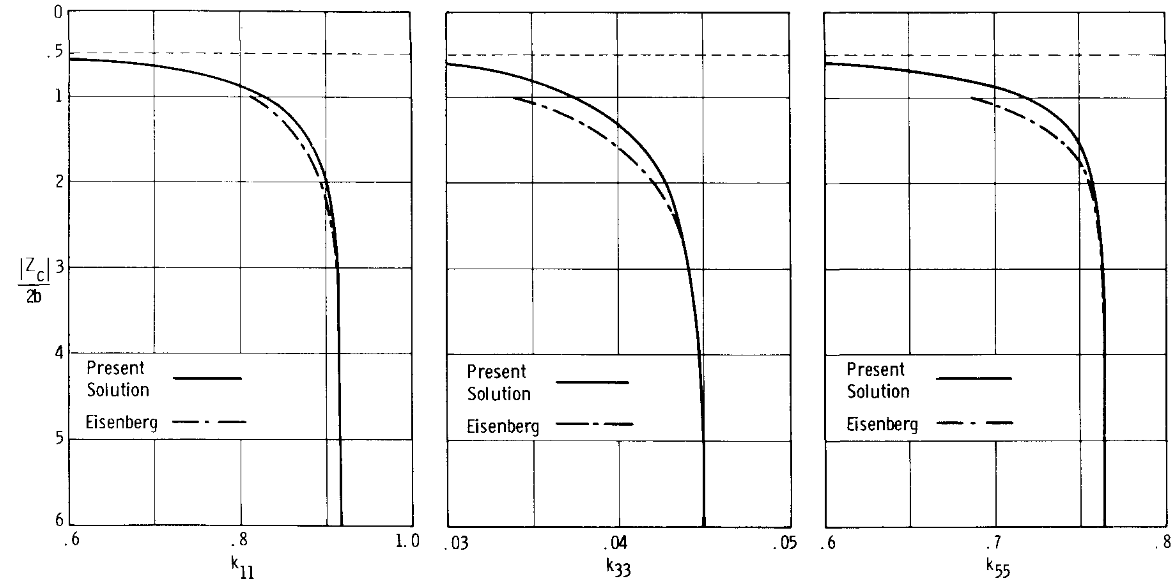

Fig. 4 Added-mass and added-moment-of-inertia coefficients for a 6:1 spheroid with the major axis parallel to the free surface.

V. Summary and Conclusions

When a body is moving near a free surface, it experiences additional hydrodynamic forces and moments due to the free surface. In the present work, the forces and moments are derived by applying the Lagally theorem for an unsteady flow, generated by doublet distributions along the axis of the body. These distributions take the presence of the free surface into account, and are given by the Munk slender-body formula, refined by an inertia correction. The hydrodynamic forces and moments are expressed in terms of the body's kinematic quantities and hydrodynamic coefficients, in which the effects of the free surface and the body geometry are embodied.

In the case of a 6:1 spheroid and a particular body of revolution in planar motion, numerical solutions for the hydrodynamic coefficients show that the free surface has negligible effects for submergence depths greater than 2 body lengths. These effects, however, increase as the body moves towards the free surface, and become prominent when the depth is less than 1 body-length. When the spheroid is at sufficiently large depths, only 6 of the 27 hydrodynamic coefficients are nontrivial; but when it is near the free surface, most or all of the 27 coefficients are nontrivial depending on the orientation of the body relative to the free surface. The added-mass coefficients deduced from the present results compare well with Eisenberg's approximate solutions for large Froude numbers.

Appendix 1 Hydrodynamic Effects of Body Rotation

In Ref. 17, it was shown that all the added-mass coefficients of a body moving through a fluid could be expressed simply in terms of singularities within the body, except those due to pure rotation. Consequently, it was not possible to express the hydrodynamic moment on the body for unsteady flow entirely in terms of actions on these singularities. It was also shown that, for ellipsoids, both the added mass for rotation and the moment can be expressed exactly in terms of the interior singularities. This result was applied to derive approximate expressions for these quantities in terms of singularities for elongated bodies.

We shall begin with the exact relation for a spheroid, e.g., Eq. (43) of Ref. 17, and show that its simplier approximate version differs negligibly from it.

The exact added-mass relation for A_{55} for a spheroid is

$$A_{55}\frac{B_{55}}{B'_{55}} + B'_{55} = -4\pi\rho\int_{z1}^{z2} m_R z dz \tag{42}$$

where

$$B_{55} = \rho\int_{V}(x^2+z^2)\mathrm{d}\tau = \pi\rho\left(H_1 + \frac{1}{4}H_2\right)$$

$$B'_{55} = \rho\int_{V}(x^2-z^2)\mathrm{d}\tau = -\pi\rho\left(H_1 - \frac{1}{4}H_2\right)$$

$$H_1 = \int_{z1}^{z2} z^2 r^2(z)\mathrm{d}z, \qquad H_2 = \int_{z1}^{z2} r^4(z)\mathrm{d}z \tag{43}$$

For an elongated spheroid, we have $\sigma = H_2/H_1 << 1$. Also, put $A_{55} = K_R B_{55}$, Eq. (42) then becomes

$$A_{55}\frac{1+\frac{\sigma}{4}}{1+\frac{\sigma}{4}} + B_{55}\frac{1-\frac{\sigma}{4}}{1+\frac{\sigma}{4}} \doteq A_{55} + B_{55} - \frac{\sigma}{2}(1-k_R)B_{55}$$
$$\doteq 4\pi\rho\int_{z1}^{z2} m_R z\mathrm{d}z$$

in which higher powers of σ have been neglected. Since $1-k_R$ is small, the term containing $\sigma(1-k_R)$ may be neglected, and we obtain the simpler form

$$A_{55} + B_{55} = 4\pi\rho\int_{z1}^{z2} m_R z\mathrm{d}z \tag{44}$$

the relation used in the text. Now, assuming

$$m_R(z) = \tfrac{1}{4}(1+k'_R)zr^2(z) \tag{45}$$

we obtain from Eq. (44)

$$1+k'_R = \left(1+\frac{\sigma}{4}\right)(1+k_R) \tag{46}$$

in agreement with Eqs. (10) and (11).

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